A Sequential Decoding Medium Rate Performance Model

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An approximate analysis of the effect of a noisy carrier reference on the performance of sequential decoding is presented. The analysis uses previously-developed techniques for analyzing noisy reference performance for medium-rate uncoded communications adapted to sequential decoding for data rates of 8 to 2048 bits/s. In estimating the 10^{-4} deletion probability thresholds for Helios, the model agrees with experimental data to within a few tenths of a dB at 8 and 2048 bits/s; the greatest error is 1.5 dB at 128 bits/s.

I. Introduction

Convolutional encoding with sequential decoding is a very powerful technique for communicating at low-error probability with deep space probes. It has been used successfully with Pioneers 9 and 10, and is planned for use on Helios. Most of the performance data for this coding technique have been developed without regard to the effects of noisy reference signals in carrier or subcarrier tracking loops. These effects must be known with fair accuracy for the optimal design of telemetry links with sequential decoding. This author (Ref. 1) previously analyzed the effect of a noisy carrier reference on sequential decoding for high data rates, and presented a general discussion of the sequential decoding noisy reference problem. This article presents a model for sequential decoding noisy reference performance that is approximately valid for the intermediate data rates of greatest interest to the DSN. The material contained herein was presented at the recent Joint Helios Working Group Splinter Session at JPL on September 27 and 28, 1973.

II. Extreme Models

At extremely high or extremely low data rates, the sequential decoding noisy reference performance is relatively well behaved. At low data rates, the time-varying carrier phase error varies rapidly enough that its effect is almost completely averaged out within one symbol time; there is no correlation between carrier reference noise in adjacent symbols, and the resultant channel model is white and Gaussian, with a somewhat degraded signal power. At extremely high data rates, the carrier phase error can be assumed to be constant over an entire data frame; thus the approach followed by Lindsey (Ref. 2) for uncoded communications is appropriate. Specifically, performance is computed as conditioned on all feasible values of phase error, and then this result is averaged over the phase error probability distribution (Ref. 3). Ironically, the ranges of data rates for which either of these extreme models fit with exactitude is outside the range of data rates supported by the DSN. If attention is restricted to low-probability long computations (erasure causing events), the validity of the high-rate model extends down to perhaps 10^3 bits/s (Ref. 1).

III. Medium Rate Model Problems

Sequential decoding data rates between 10 and 10^3 bits/s must be categorized as medium data rates from the performance modeling standpoint. Their performance lies somewhere between the performance predicted by the high- and low-rate models. But, as observed by Stolle and Dolainsky (Ref. 4), there is a "lot of in-between." There are two primary difficulties associated with establishing an accurate performance model for these medium data rates. The first is that we do not really know over what interval of data record the sequential decoding search is defined. It could be argued that the computation distributions for most long searches are defined by a noisy "barrier" of perhaps 3 to 10 bits in length. However, the backward search depth in sequential decoding is on the order of a constraint-length, or two, and the noise at each symbol encountered within a search must necessarily affect the number of computations needed in that search. Finally, we note that individual searches can interact up to the limit of the frame length, where they are forcibly terminated. None of these correctly represents the effective memory duration of the decoder, yet all are partially correct. The second problem is that the carrier reference errors interact with the record length that defines the searches. For example, the data rate/loop bandwidth ratio δ may be such that the carrier phase reference is essentially constant over the 1 to 3 bit times that define most short searches; yet when a large carrier reference phase error occurs a long search results, with a number of computations, which is dependent not only upon the section of data over which the carrier reference is poor, but upon a long preceding section of data within which the carrier reference varies significantly. From these considerations, I do not expect that the noisy reference performance of sequential decoding can be accurately modeled with any simple technique.

IV. Medium Rate Models

Fairly tractable techniques exist for calculating the performance of uncoded telemetry at medium data rates (Ref. 5). They exist because the error probability in uncoded telemetry depends uniformly upon the signal, noise, and carrier reference statistics over a predetermined interval of the data signal, and not at all outside that interval. The approach I have been following in modeling the sequential decoding performance at medium data rates has been to use an uncoded medium rate technique to extend the validity of the high-rate model to lower data rates. This will be discussed shortly. Massey (Ref. 6) has suggested that the problem could be approached from the low-rate end, by modeling the channel statistics, and the associated Pareto exponent, using the medium rate uncoded channel techniques. The effect of carrier reference perturbations which are correlated from symbol to symbol can then be added as a perturbation term. This is quite clearly the best approach at 8 bits/s where the correlation between even adjacent symbols is relatively small. It may prove difficult with this technique to get above 30 to 50 bits/s, where significant correlations of the carrier reference error exist over more than 4 bit times.

Extension of the high-rate model into the medium-rate region involves a number of assumptions and approximations: (1) the decoding computation distribution depends predominantly upon isolated long searches that are defined in structure over some fixed length segment of the data record, called T_m; (2) the computation distribution for long searches depends uniformly upon the carrier reference noise throughout the T_m interval; (3) the correlation between T_m intervals within a frame is independent of their position within that frame. The analysis technique implied by these assumptions is as follows: the channel signal-strength statistics are computed for the signal averaged over the T_m interval using the techniques for uncoded telemetry (Ref. 5). The sequential decoding performance is computed conditioned upon the channel signal strength, and then averaged over the distribution of the channel signal strength. Specifically we compute

$$P_{\text{deletion}} = \int_0^2 \Pr\left\{\text{deletion} \,\middle|\, \text{No. of computations, SNR} = \frac{E_b}{N_0}(1 - x/2)^2\right\} h(x)\,dx + \int_2^\infty h(x)\,dx \qquad (1)$$

where

$$
\left.\begin{aligned}
h(x) &= \sqrt{\frac{a}{\pi}}\, e^{\sqrt{2ab}} (\rho_L' x)^{-1/2} \exp\{-a\rho_L' x - b/(\rho_L' x)\}\, \rho_L' \\
\rho_L' &= \text{effective loop signal-to-noise ratio (SNR)} \\
a &= \frac{B(\delta)}{4} (1 + \sqrt{1 + 4/B(\delta)}) \\
b &= a - 1 + 1/4a \\
\delta &= 1/W_L T_m \\
B(\delta) &= 1/[\delta - (\delta^2/4)(1 - e^{-4/\delta})]
\end{aligned}\right\} \tag{2}
$$

This approximate distribution has been developed for analysis of medium-rate uncoded communications (Ref. 5), and produces more consistent results than previous medium-rate theory for uncoded communications (e.g., Refs. 7, 8). The effective loop signal-to-noise ratio ρ_L', is determined parametrically by $\rho_L = \rho_L' \exp\{1/(2\rho_L')\}$, where ρ_L is the true carrier loop SNR in the operating bandwidth, as computed by Lindsey (Ref. 7), and includes the effects of bandwidth expansion of the limiter-phase-locked loop.

The Pr {deletion | No. of computations, SNR} is well-approximated by

$$
\Pr\{C_L > N \cdot L\} = \exp \sum_{\substack{n=-1,1 \\ r=0,2}} \{A_{n,r} R^r (\ell n N)^n\} \tag{3}
$$

where C_L is the number of computations per frame, R is bit SNR, N is the average number of computations-per-bit within a frame of length L, and $\{A_{n,r}\}$ is a coefficient matrix determined by a two-dimensional least-squares fit to experimental data. The computation distribution for the Helios frame of 1152 bits has been experimentally determined by Dolainsky (Ref. 9). Sampled points from this experimental work appear as points on Fig. 1. The coefficients $\{A_{n,r}\}$ corresponding to these data appear in Table 1. The solid lines of Fig. 1 show this approximation. The remainder of the data presented in this article were computed from these simulation data. They deviate typically by only a few tenths of a dB from data computed for the Pioneer 192-bit frame length (Ref. 1, Table 3).

The performance computed by this technique is significantly dependent upon the value of T_m used. The true value is, as noted above, unknown. It is clear that the effective value for T_m depends upon the number of computations in each search. If the number of computations-per-bit is very small, then all decoding decisions are made on the basis of very short pieces of the received data, and $T_m = 1$ is appropriate. On the other hand, when the number of computations-per-bit is large, at least some of the decoding decisions must be made over long segments of the received data, and T_m may be much greater than one. For the purposes of evaluating the dependence of T_m upon number of computations, the decoding search process has been roughly modeled as a full tree of "b" branches per node. The average search length, T_m, within a search of N steps is thus approximated as the average branch length in a tree of N edges.

$$
T_m \approx \left(\frac{b}{b-1}\right)\left(1 - \frac{1}{N} \ell n \left[1 + N\frac{b-1}{b}\right] \Big/ \ell n\,[b]\right) \cdot T_b \tag{4}
$$

In Eq. (4), T_b is bit duration. This expression is strictly valid only for integer $b > 1$, and for $N = b^\ell$, all ℓ. It should be used for other values of b and N only with the caveat that it may be a very rough approximation. The sequential decoding tree can have at most two branches per node; numerical results in the remainder of this article have been computed with $b = 2$. It would, of course, be much more correct to compute the numerical sequential decoding model using the true joint distribution of T_m and N, but such would require much more computing time than is used by the current model, and the needed statistics are not currently available.

V. Modeled Decoding Performance

The modeled decoding performance estimate is perhaps best displayed graphically. Figures 2a and 2b show the deletion probability as a function of the total-power-to-noise-density ratio (P_T/N_0) for the Helios modulation indices (MI) and data rates. A 12-Hz carrier-tracking loop is assumed in the DSN receiver. Figures 3a through 3c show the modeled computational distribution function for 2048 bits/s, for three power levels that differ by 0.5 dB, and for several modulation indices. A comparison of these curves with real-time[1] and nonreal-time (Ref. 10) experimental results reveals that the modeled curves agree, within experimental tolerances, with the experimental curves. Figures 4a through 4c show the modeled computation distribution function for 128 bits/s. Here, modeled and experimental results agree suitably for low

[1]Real-time tests of sequential decoding performance in Helios configuration have been initiated at DSS 71, and will be reported in a future issue.

modulation indices, but disagree at the high modulation indices where the noisy carrier reference is the dominant effect. For modulation indices of 50 to 60 deg, the model predicts significantly more degradation than is observed experimentally. Figures 5a through 5c show the modeled computation distribution function for 8 bits/s. Again, at this low data rate, the model agrees with the experiment to within experimental tolerance.

Figure 6 (solid lines) shows the modeled threshold total-power-to-noise density ratio required to achieve a 10^{-4} deletion probability for the Helios data rates and modulation indices. The corresponding experimentally determined threshold (Ref. 10) is also shown for comparison. Model inaccuracies at 128 bits/s have resulted in a statistically-significant 1.5-dB separation between model computations, and extrapolation of the experimental data at this data rate.

VI. Commentary and Future Work

At this point, modeling of the sequential decoding noisy reference performance by the techniques described here appears to be at, or near, a dead-end. The results are close, and perhaps useable for system design, but they are not exact. There is no obvious physically-justifiable change to the modeling technique or parameters that can be applied with assurance of improving the result, or of representing more exactly the physical process of sequential decoding.

The choice of T_m is perhaps the weakest link within the model. T_m has been used by Stolle (Ref. 11) as a free parameter to manipulate a model similar to the one presented here[2] into agreement with experimental data. The approach is successful, and clearly a good one for improving the model with experimental data. However, the effective T_m/T_b ratio determined this way is largest at 128 bits/s, and the physical interpretation of that fact is not clear at this time. A physical explanation of this behavior may appear after the technique has been applied to a large data set.

Two approaches are known at this time that may be followed to develop an improved model of sequential decoding noisy reference performance. Both represent greater computational work than the present model. One approach is channel modeling (Ref. 6). The other is to use the joint distribution of N and T_m, and integrate over T_m as well as over the noisy reference loss distribution at fixed T_m. For this to work, we must experimentally determine the joint distribution, or hypothesize a form for it if the performance results are suitably insensitive to the detailed form of the distribution. Neither approach is assured of success, and, until success is obtained by an alternate technique, the model approach described here, with its known biases, provides a useful (conservative) approximation to sequential decoding performance in the DSN.

[2]E. Stolle's noisy reference model basis corresponds to Eqs. 4 and 5 of Ref. 5, and Eq. 3, Table 3 of Ref. 1.

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Table 1. $A_{n,r}$ for Helios frame

r \ n	-1	0	1
0	2.397	8.824	-0.9887
1	-0.5331	-6.788	1.569
2	0.02303	0.8848	-0.8543

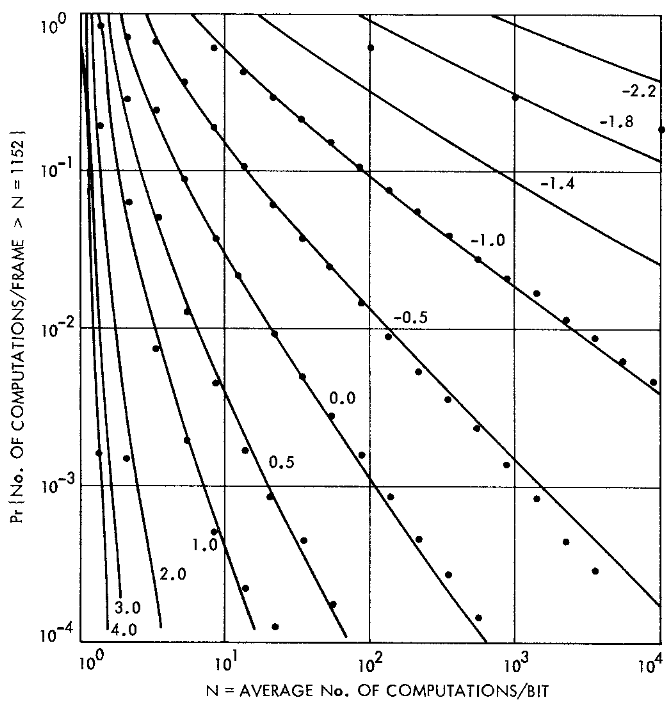

Fig. 1. Distribution of computations for sequential decoding of Helios frame

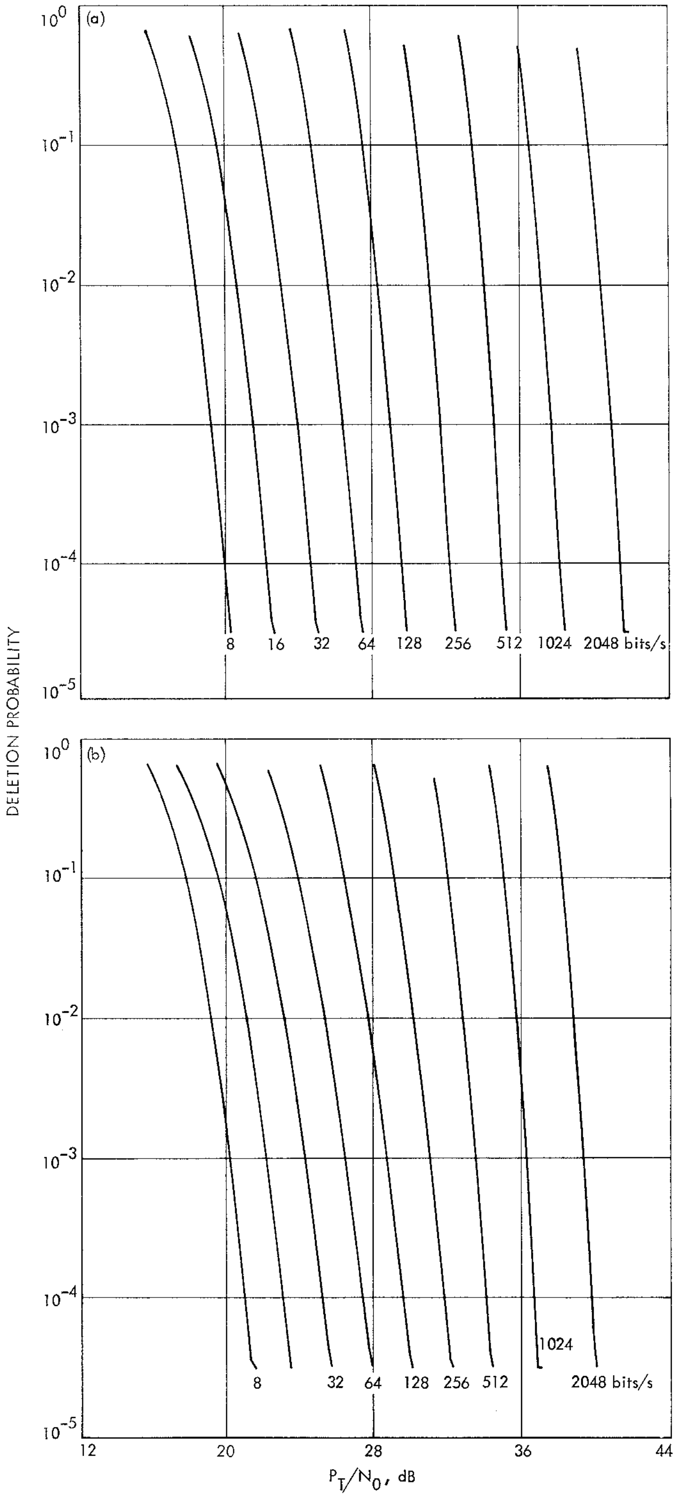

Fig. 2. Modeled deletion probability for Helios; (a) MI = 42 deg, (b) MI = 55 deg

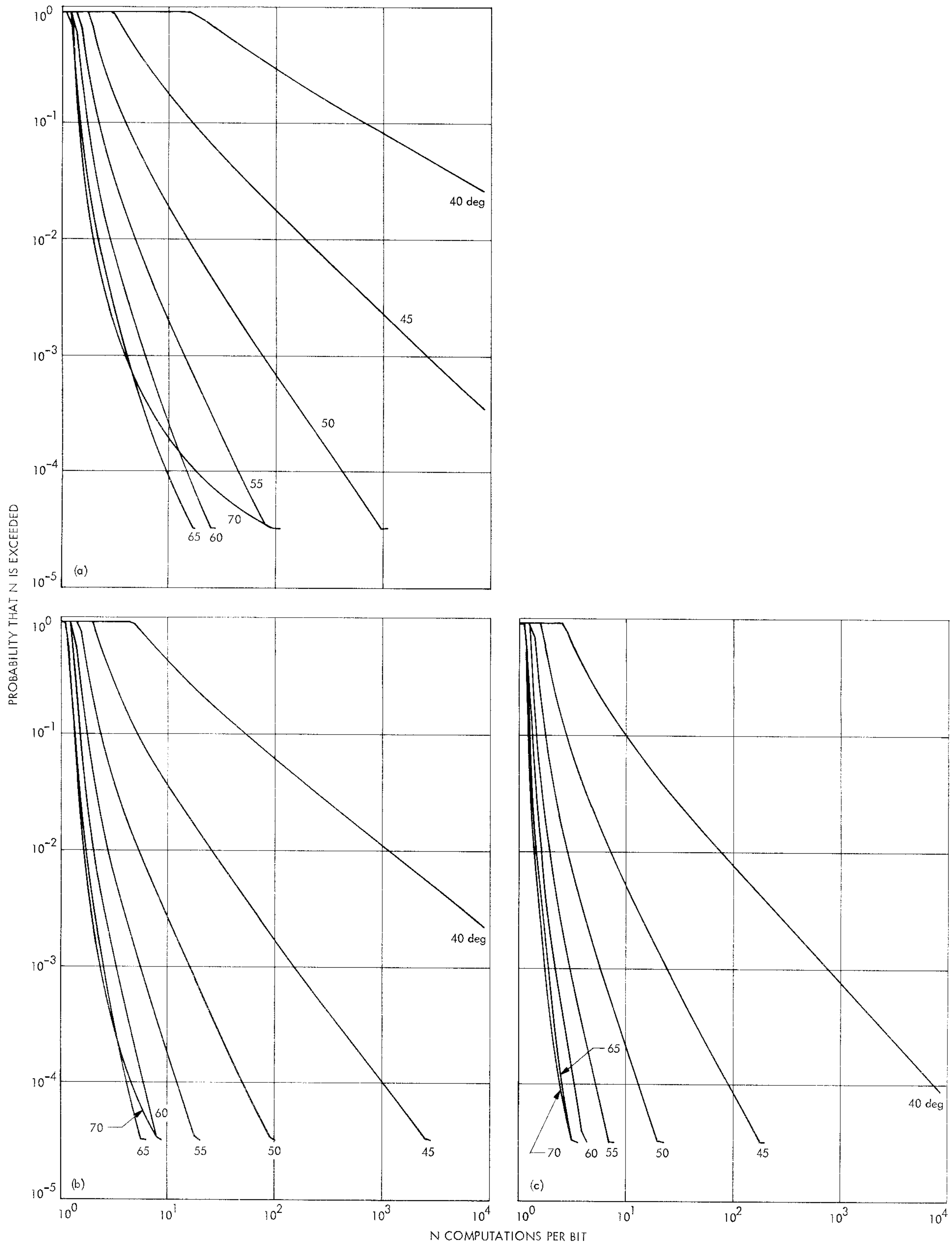

Fig. 3. Modeled distribution of computations for Helios decoding at 2048 bits/s; (a) P_T/N_0 = 39.1 dB, (b) P_T/N_0 = 39.6 dB, (c) P_T/N_0 = 40.1 dB

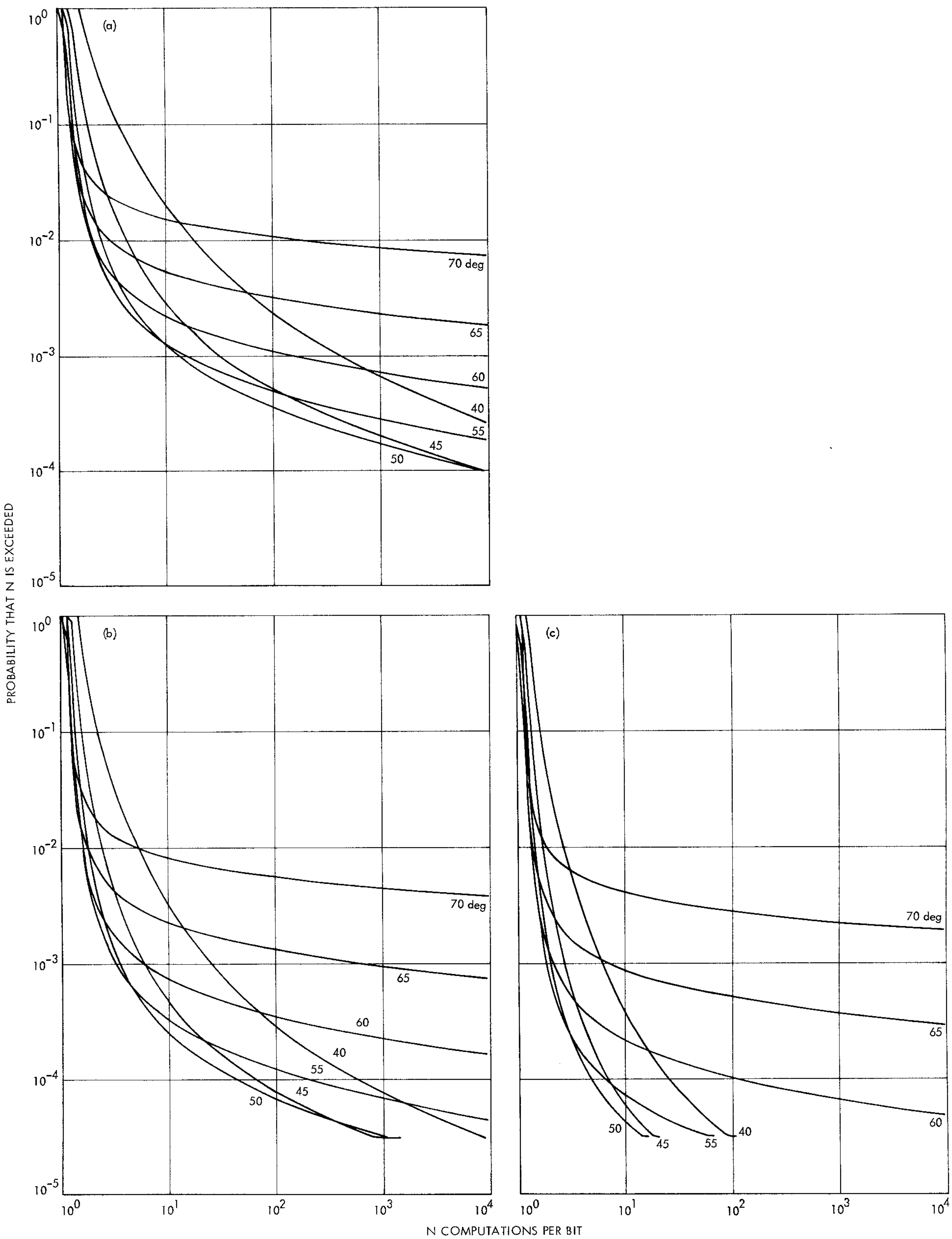

Fig. 4. Modeled distribution of computations for Helios decoding at 128 bits/s; (a) P_T/N_0 = 29.0 dB, (b) P_T/N_0 = 29.5 dB, (c) P_T/N_0 = 30.0 dB

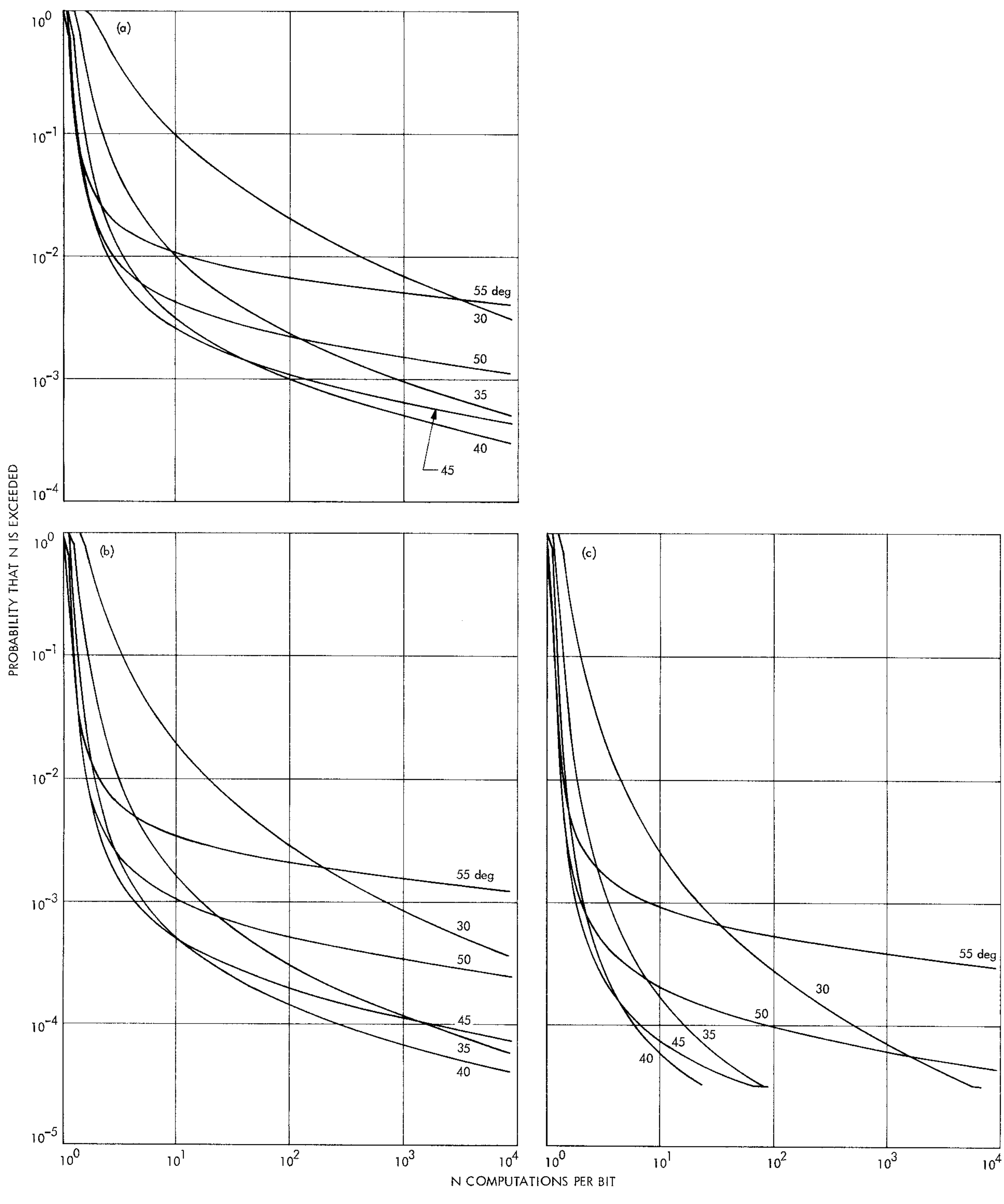

Fig. 5. Modeled distribution of computations for Helios decoding at 8 bits/s; (a) $P_T/N_0 = 19.5$ dB, (b) $P_T/N_0 = 20.0$ dB, (c) $P_T/N_0 = 20.5$ dB

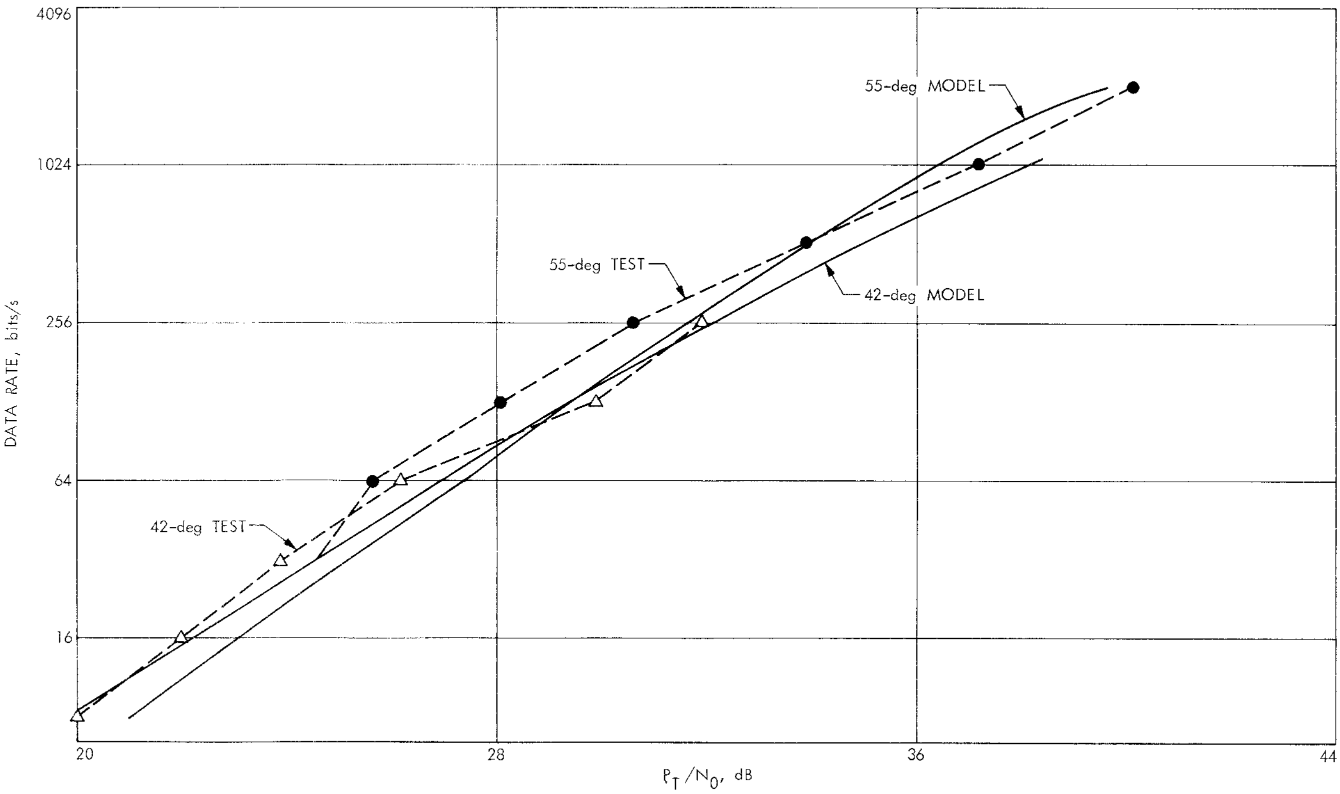

Fig. 6. P_T/N_0 thresholds for Helios for 10^{-4} deletion probability; comparison of model and extrapolated experiment